Project Number: 2018-1-IT02-KA201-048274

INTERVENTION TOOL

Constructing the Meaning of Variable and of Expression in One Variable

University of Genova[1]

1. Introduction

In order to develop educational activities aimed to support memory in arithmetic, we refer to some theoretical frameworks that will be described in the session 2.
In session 3 the design of the educational activities is described. In particular, if the activities are addressed to a student or the class, the educational aim of the activities, the Cognitive area and math domain of interest and the Mathematical objects in the areas of difficulties identified through the B2 questionnaire

2. Theoretical framework of reference

The theoretical references that helped us to design the following activities are:
1) **Universal design for learning (UDL) principles** (Table 3), a framework specifically conceived to design *inclusive* educational activities (http://udlguidelines.cast.org/)

Table 3: UDL guidelines

	Provide multiple means of **Engagement**	Provide multiple means of **Representation**	Provide multiple means of **Action & Expression**
	Affective Networks The "WHY" of learning	Recognition Networks The "WHAT" of learning	Strategic Networks The "HOW" of learning
Access	Provide options for **Recruiting Interest** (7) • Optimize individual choice and autonomy (7.1) • Optimize relevance, value, and authenticity (7.2) • Minimize threats and distractions (7.3)	Provide options for **Perception** (1) • Offer ways of customizing the display of information (1.1) • Offer alternatives for auditory information (1.2) • Offer alternatives for visual information (1.3)	Provide options for **Physical Action** (4) • Vary the methods for response and navigation (4.1) • Optimize access to tools and assistive technologies (4.2)
Build	Provide options for **Sustaining Effort & Persistence** (8) • Heighten salience of goals and objectives (8.1) • Vary demands and resources to optimize challenge (8.2) • Foster collaboration and community (8.3) • Increase mastery-oriented feedback (8.4)	Provide options for **Language & Symbols** (2) • Clarify vocabulary and symbols (2.1) • Clarify syntax and structure (2.2) • Support decoding of text, mathematical notation, and symbols (2.3) • Promote understanding across languages (2.4) • Illustrate through multiple media (2.5)	Provide options for **Expression & Communication** (5) • Use multiple media for communication (5.1) • Use multiple tools for construction and composition (5.2) • Build fluencies with graduated levels of support for practice and performance (5.3)
Internalize	Provide options for **Self Regulation** (9) • Promote expectations and beliefs that optimize motivation (9.1) • Facilitate personal coping skills and strategies (9.2) • Develop self-assessment and reflection (9.3)	Provide options for **Comprehension** (3) • Activate or supply background knowledge (3.1) • Highlight patterns, critical features, big ideas, and relationships (3.2) • Guide information processing and visualization (3.3) • Maximize transfer and generalization (3.4)	Provide options for **Executive Functions** (6) • Guide appropriate goal-setting (6.1) • Support planning and strategy development (6.2) • Facilitate managing information and resources (6.3) • Enhance capacity for monitoring progress (6.4)
Goal	**Expert Learners** who are... **Purposeful & Motivated**	**Resourceful & Knowledgeable**	**Strategic & Goal-Directed**

[1] Emanuela De Negri, Elisabetta Robotti, Francesca Morselli, Paola Viterbori, Anna Siri, Laura Capelli

The Center for Applied Special Technology (CAST) has developed a comprehensive framework around the concept of Universal Design for Learning (UDL), with the aim of focusing research, development, and educational practice on understanding diversity and facilitating learning (Edyburn, 2005). UDL includes a set of Principles, articulated in *Guidelines and Checkpoints*[2]. The research grounding UDL's framework is that "learners are highly variable in their response to instruction. [...]"
Thus, UDL focus on these individual differences as an important element to understanding and designing effective instruction for learning.
To this aim, UDL advances three foundational Principles: 1) provide multiple means of representation, 2) provide multiple means of action and expression, 3) provide multiple means of engagement. In particular, guidelines within the first principle have to do with means of perception involved in receiving certain information, and of "comprehension" of the information received. Instead, the guidelines within the second principle take into account the elaboration of information/ideas and their expression. Finally, the guidelines within the third principle deal with the domain of "affect" and "motivation", also essential in any educational activity.
For our analyses we will focus in particular on specific guidelines within the three Principles[3].
In order to characterize students' difficulties in geometry, we refer to the following elements of Karagiannakis' and colleagues' frame (Table 1), which dealt with Memory in retrieval of geometrical facts and geometrical processing: retrieval geometrical facts, remembering theorems, remembering hypothesis and thesis which are focusing on.

Table 1: Karagiannakis's and colleagues' frame: domains of the four-pronged model and sets of mathematical skills associated with each domain

Domain	Mathematical skills associated with the domain
Core number	Estimating accurately a small number of objects (up to 4); estimating approximately quantities; placing numbers on number lines; managing Arabic symbols; transcoding a number from one representation to another (analogical-Arabic-verbal); counting principles awareness
Memory (retrieval and processing)	Retrieving numerical facts; decoding terminology (numerator, denominator, isosceles, equilateral); remembering theorems and formulas; performing mental calculations fluently; remembering procedures and keeping track of steps
Reasoning	Grasping mathematical concepts, ideas and relations; understanding multiple steps in complex procedures/algorithms; grasping basic logical principles (conditionality – "if … then …" statements – commutativity, inversion); grasping the semantic structure of problems; (strategic) decision-making; generalizing
Visual-spatial	Interpreting and using spatial organization of representations of mathematical objects (for example, numbers in decimal positional notation, exponents, geometrical 2D and 3D figures or rotations); placing numbers on a number line; confusing Arabic numerals and mathematics symbols; performing written calculation when position is important (e.g. borrowing/carrying); interpreting graphs and tables

FaSMEd activities are organized in sequences, that encompass group work on worksheets and class discussion where selected group works are discussed by the whole class, under the orchestration of the teacher. Taking into account formative assessment strategies and technology functionalities, Cusi, Morselli & Sabena (2017, p. 758) designed three types of worksheets for the classroom activity:

> "(1) *problem worksheets:* worksheets introducing a problem and asking one or more questions involving the interpretation or the construction of the representation (verbal, symbolic, graphic, and tabular) of the mathematical relation between two variables (e.g. interpreting a time-distance graph);
>
> (2) *helping worksheets*, aimed at supporting students who face difficulties with *the problem worksheets* by making specific suggestions (e.g. guiding questions);
>
> (3) *poll worksheets*: worksheets prompting a poll among proposed options".

[2] For a complete list of the principles, guidelines and checkpoints and a more extensive description of CAST's activities, visit http://www.udlcenter.org
[3] The items are taken from the interactive list at http://www.udlcenter.org

The authors identified feedback strategies (Table 5) the teacher may adopt to give feedback to students (Cusi, Morselli & Sabena, 2018, p. 3466). These strategies are employed in the class discussion that is organized by the teacher after the group work on worksheets.

Table 5:

Revoicing	When the teacher mirrors one student's intervention so as to draw the attention on it. Often, during the revoicing, the teacher stresses with voice intonation some crucial words of the sentence she is mirroring. Rephrasing takes place when the teacher reformulates the intervention of one student, with the double aim of drawing the attention of the class and making the intervention more intelligible to everybody.
Rephrasing	Rephrasing takes place when the teacher reformulates the intervention of one student, with the double aim of drawing the attention of the class and making the intervention more intelligible to everybody. Rephrasing is applied when the teacher feels that the intervention could be useful but needs to be communicated in a better way so as to become a resource for the others. […] The revoicing and rephrasing strategies […] turn one student (the author of the intervention) into a resource for the class.
Rephrasing with scaffolding	When the teacher, besides rephrasing, adds some elements to guide the students' work.
Relaunching	When the teacher reacts to a student's intervention, which (s)he considers interesting for the class, not giving a direct feedback, but posing a connected question. In this way, by relaunching the teacher provides an implicit feedback […] on the student's intervention, suggesting that the issue is interesting and worth to be deepened or, conversely, has some problematic points and should be reworked on.
Contrasting	Contrasting takes place when the teacher draws the attention on two or more interventions, representing two different positions, so as to promote a comparison. By contrasting, […] the authors of the two positions may be resource for the class as well as responsible of their own learning.

We draw from the FaSMEd experience the idea of creating classroom activities in the formative assessment perspective, which may promote inclusion.

3. Design

3.1 Difficulties identified through the B2 questionnaire

We detect difficulties in the following item of B2:

If a=3 what is the value of 2a+1?

If x= -4, what is the value of 24/x?

These difficulties are related to the construction of the meaning of variable and of expression depending on such a variable.

3.2 Cognitive area and math domain of interest

The area of difficulties identified through the B2 questionnaire is related to the domain of *Algebra*. In particular, the difficulties are related to the construction of the meaning of variable and of expression depending on such a variable. Thus, *Reasoning* is the cognitive area involved (Table 1).

Table 1: The difficulties detected are linked to the cognitive domain of *Reasoning* and in the domain of *Algebra*

	Arithmetic	Geometry	Algebra
Memory			
Reasoning			If a=3 what is the value of 2a+1? If x= -4, what is the value of 24/x?
Visuo-spatial			

3.3 Educational Aims

The intervention tool is aimed at *Constructing the Meaning of variable and of expression in one variable.*

3.4 Addressing to Student /class

The Intervention tool is articulated in a set of activities that have to be carried out with all the class, in a perspective of inclusion.

3.5 Educational activities: the Intervention Tool

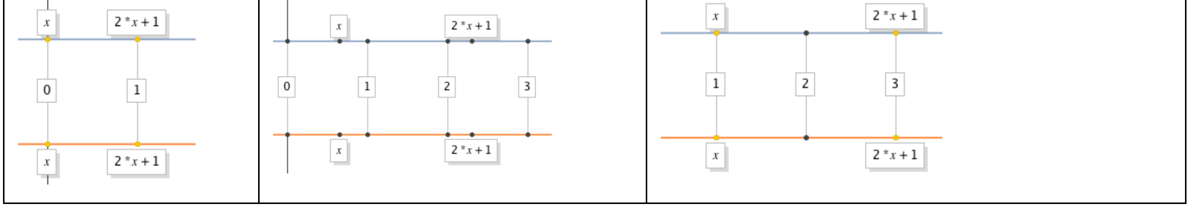
The teaching sequences are conceived to address specific learning difficulty, within an inclusive perspective. The activities are not to be intended as mere exercises. Instead they play the role of cognitive training. In cognitive training the student is led to perform a series of exercises that are focused in the same mathematical content. In order to have a repeated sequence, ICT are used.

Dynamic representation of variable and expression depending on such a variable.

The first idea in designing activities relies on the use of the software AlNuSet, (see http://www.alnuset.com/en/alnuset). AlNuSet was designed for secondary school students (from age 12-13 to age 16-17) and it is made up of three separate environments that are tightly integrated: the Algebraic Line, the Algebraic Manipulator, and the Cartesian Plane. We will describe the features of the Algebraic Line, through the following activity[4], which support the conceptualization of algebraic notions of variable and expression depending on a variable in MLD students (Robotti, E. 2016; Robotti E., Baccaglini-Frank A., 2017).

On the Algebraic Line it is possible to place variables and expressions that depend from them. To do this, the user has to type a letter, for example, “x”, and a mobile point will appear on the line. The point can vary within the chosen set of numbers (natural, whole, rational, or real[5]) and variation can be controlled directly by the user through dragging. This feature was designed so that important aspects of the notion of *variable* could become embodied. Moreover, it is possible to construct expressions on the line that depend on a chosen variable, for example, 2x+1. This dependent expression cannot be acted upon directly, but it will move as a consequence when x is dragged. The dependent expression will assume the positions on the line that correspond to the values it takes on when the dependent variable takes on the value it is dragged to (Figure 1).

Figure 1. The movement of the variable x on the Algebraic Line produces the movement of the dependent expression 2x+1 on the line.



[4] For a more detailed description of these environments see www.alnuset.com

[5] Of course the representations of the numerical sets are accomplished on a computer, so the sets are actually finite and discrete, but they simulate – with some limitations – the properties of the number sets they represent.

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We note that the described functionalities propose different representations (UDL Principle 1) and they are designed to foster for the user a mediation of the algebraic concepts of *variable* and *dependent expression,* through a dynamic model that can be acted upon (UDL Principle 2). The mediation can occur thanks to visual and kinaesthetic channels, without the need of visual verbal means (written language). The construction of the concept realized as so may allow students, and especially students with MLD, to find mnemonic references that are appropriate for their cognitive style. This allows them to start using representations of the fundamental algebraic concepts at stake, and possibly to place and retrieve them from long term memory in a more effective way.

With the support of AlNuSet, the teacher can promote a discussion among the students of the class in order to conceptualize the idea of *variable*.
As matter of fact, he/she can ask to the students to move *x* along the line and to answer the following questions: “What can you observe?”, “How do you interpret what happens?”

Moreover, the teacher can promote also a discussion among the students in order to conceptualize the idea of *expression depending on the variable x*.
Therefore, the teacher asks to the students to digit *2x+1* in editor space of the Algebraic Line and he/she launch a discussion by the following question: “What happens on the Algebraic Line?
“How do you interpret what happens to the algebraic expression 2x+1?”.

It could be interesting, in a first time, promoting the definition of an hypothesis without the dynamic support of AlNuSet.
Thus, the teacher could ask to the students: “If x=3, what do you think will be the value of the expression 2x+1? Make your hypothesis, compare it with your schoolmates and then verify it on the Algebraic Line of AlNuSet”.

A discussion (guided by the teacher) about what students observe on the Algebraic Line and how they can interpret it in algebraic way, allows students to construct the meaning of *variable* and of *expression depending on such a variable*.

In terms of formative assessment, strategy 2 (engineering classroom discussions) is activated. During the discussion, strategies 5 and 4 are activated, since students may intervene to express their doubts (thus becoming owners of their own learning) or to give explanations to their mates (thus becoming resources for the mates). The teacher and the peers may provide feedback to a student, thus activating strategy 3.

Representation of the relation between variable and expression depending on such a variable on a Cartesian plan and on a table

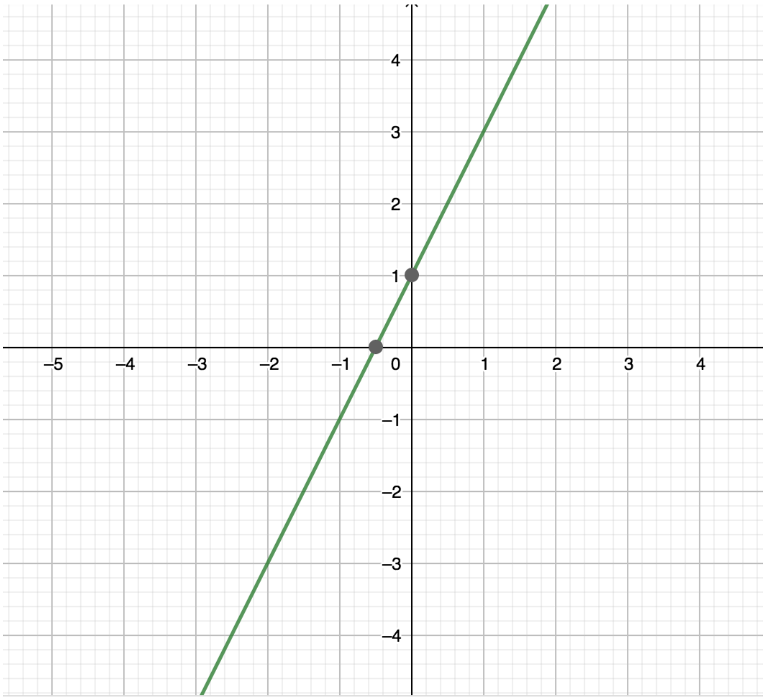
We consider a table defining the relation between the variable “x” and the expression 2x+1.

x	**2x+1**
1	
2	
3	
0	
-1	
-4	

The teacher asks to the students to calculate the value of the expression 2x+1 starting from the values of the independent variable “x”

x	2x+1
1	2*1+1=2+1=3
2	2*2+1=4+1=5
3	2*3+1=6+1=7
0	2*0+1=0+1=1
-1	2*(-1)+1=-2+1=-1
-4	2*(-4)+1=-8+1=-7

The teacher asks students to draw the relation on the Cartesian plane:



The teacher guides the discussion about the relation between *x* and the expression *2x+1* both through geometrical representation (on the Cartesian plan) and the algebraic relation (on the table) so that students will be able to pass from a code to the other one (transcoding process).

Concrete representation of a variable and of an expression depending on such a variable

The teacher presents two identical boxes (each represents *x*) and 1 straw (the constant), (Figure 2). By varying the number of straws in the boxes (the same for both, this means varying the value of the variable), the total straws vary (varying the value of the expression depending on such a variable).

Figure 2: Varying the value of the expression 2x+1 by varying the number of straws in the boxes (x)

2x+1 (if x=0, 2x+1is 1)

if x=1, 2x+1 is 3

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 if x=3, 2x+1 is 7	 if x=4, 2x+1 is 9

The meaning of “variable” and of “expression which depends on such a variable” in algebra is constructed in a perceptive way by the manipulation of concrete objects.

Discussion through UDL guidelines about the above-mentioned activities

We observe that the same educational aim of constructing the meaning of “variable” and of “expression depending on such a variable” in algebra is approached in different ways by acting on the three principles of UDL (Table 7, in *red* our comments to illustrate the connection between the principles and our activities).

Table 7: Analysis of the activities through the Table of UDL principles.

Engagement	***Representation***	***Action & Expression***
Recruiting interest Optimize individual choice and autonomy Optimize relevance, value, and authenticity Minimize threats and distractions	***Perception*** Offer ways of customizing the display of information Offer alternatives for auditory information Offer alternatives for visual information Different registers through which information are displayed (visual-dynamic; visual; symbolic)	***Physical Action*** Vary the methods for response and navigation Optimize access to tools and assistive technologies
Sustaining effort Persistence Heighten salience of goals and objectives Vary demands and resources to optimize challenge Foster collaboration and community Increase mastery-oriented feedback Vary demands and resources to optimize challenge Foster collaboration and community	***Language & Symbols*** Clarify vocabulary and symbols Clarify syntax and structure Offer alternative language and symbols to decode information and to work on the information Support decoding of text, mathematical notation, and symbols *This is promoted by the dynamic action, and by the manipulation of objects* Promote understanding across languages	***Expression Communication*** Use multiple media for communication Use multiple tools for construction and composition Build fluencies with graduated levels of support for practice and performance To use different registers in order to communicate *This is promoted by the use of terms that are alternative to the formal ones to speak about mathematical objects. Such alternative terms recall*

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Oriented feedbacks support engagement and motivation with respect the elaboration of the solution of the task	Illustrate through multiple media *This is promoted by the activities of transcoding among different register of representation* Support decoding of text, math notation and symbols *This is promoted by the visualization of different registers at the same time (for example, on the Algebraic Line, a variable is a mobile point on the line and it is labelled by x)*	*the meaning that was constructed by the students. For instance, students who worked with AlNuSet are keen to speak of "moving point" when they refer to the variable.* *Moreover, in the activities virtual or concrete mathematical manipulatives are provided. For instance, dragging a moving point may help visualizing that the variable may have different values on the number line.* *Some activities that are connected to this principle are:* *- asking to read a table using AlNuSet (to transcode from table to AlNuSet)* *-asking to read AlNuSet with a table (to transcode AlNuSet into table)*
Self Regulation Promote expectations and beliefs that optimize motivation Facilitate personal coping skills and strategies Develop self-assessment and reflection *Formative assessment strategies, as discussed in section 2, may help self-assessment and reflection. More specifically, the teacher may provide different types of feedback*	***Comprehension*** Activate or supply background knowledge Highlight patterns, critical features, big ideas, and relationships (checkpoint 3.2) Guide information processing and visualization Maximize transfer and generalization Perception, language and symbols, comprehension (Constructing useable knowledge, knowledge that is accessible for future decision-making, depends not upon merely perceiving information, but upon active "information processing skills")	***Executive functions*** *Guide appropriate goal-setting* *The use of artefacts may also be a support for memory. Artefacts guide students' process of inquiry, providing feedback to their process.* *Support planning and strategy development* *Facilitate managing information and resources* *Enhance capacity for monitoring progress*

This allows students to construct meaning for the algebraic notions at stake.

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