Chapter 3
How to address Difficulties and Learning Disabilities in Math

Developed in the framework of the European project

SMiLD

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Introduction

This chapter is aimed at presenting and discussing the ways to address difficulties and learning disabilities in mathematics that were developed and tested during the SMILD project.

As pointed out in Chapter 1, the way of addressing difficulties and learning disabilities varies across states. What is common is the fact that the teacher plays a crucial role in identifying difficulty, understanding it and deciding how to act, with the final aim of helping the student. The SMILD project is exactly conceived to work *for the teachers and with the teachers* to help students in need. Such a way of addressing difficulties and learning disabilities is organized in two steps: at first, it is necessary to understand such difficulties, that is to say identify profiles of difficulty. This is done in the project by means of two questionnaires (B1 and B2), that are described in detail in Intellectual Output 1. Once identified the profiles of difficulty of the students, it is possible to design and implement intervention tools for a specific difficulty, that can be used by the teacher in interaction with a single student or when teaching to the whole class.

A key feature of the entire project is the fact that the design of the questionnaires and of the interventions tools is inspired by the state of the art of research and practice on mathematical learning difficulties and disabilities. Within the project we identified **theoretical tools** that may frame the design and management of the intervention tools for individual students and/or for the whole class. Such theoretical tools helped us to identify **general guidelines for the design** of efficient intervention tools. Moreover, we designed and tested **intervention tools** (Intellectual Output 2) addressing specific difficulties. The final product is a free accessible set of ICT based tools available in English plus the 3 different languages represented within the project consortium (Italian, Polish and Portuguese) in order to ensure high transferability potential of the intellectual output. An interesting feature is that each partner proposed intervention tools that were designed in reference to a specific national context, but that now, thanks to the project, can be exploited also by the teachers of other countries. Another relevant issue is the fact that the design followed a cross-national review, so that each intervention tool was improved thanks to the comments of the project partners.

This chapter contains:

- The theoretical framework that was used to design effective intervention tools. Such theoretical framework refers to inclusive education approaches (udl principles, see http://udlguidelines.cast.org/) and to formative assessment (inspired by the fasmed project, see https://research.ncl.ac.uk/fasmed/);
- The guidelines we developed for the design of the intervention tools; we point out that the guidelines should in principle also frame the design of other intervention tools by the teachers that read this chapter;
- Some examples of intervention tools that were designed and tested; we point out that more intervention tools are described in Intellectual Output 2.

The contents are developed in order to make full use of ICT and media available on-line and addressed to teaching and learning Math, providing external links to portals, websites, on-line publication, pdf documents, videos etc.

1. Designing the Intervention Tools – a Theoretical Framework

Karagiannakis's and colleagues (2016), propose a model classifying mathematical skills involved in learning mathematics into four domains: Core number, Memory, Reasoning, and Visual-spatial (the frame is presented in Table 1). Their findings support the hypothesis that difficulties in learning mathematics can have multiple originsand they provide a means for sketching students' mathematical learning profiles.

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The frame helps to characterize students' difficulties in mathematics.

Table 1: Karagiannakis's and colleagues' frame: domains of the four-pronged model and sets of mathematical skills associated with each domain

Domain	Mathematical skills associated with the domain
Core number	Estimating accurately a small number of objects (up to 4); estimating approximately quantities; placing numbers on number lines; managing Arabic symbols; transcoding a number from one representation to another (analogical-Arabic-verbal); counting principles awareness.
Memory (retrieval and processing)	Retrieving numerical facts; decoding terminology (numerator, denominator, isosceles, equilateral); remembering theorems and formulas; performing mental calculations fluently; remembering procedures and keeping track of steps.
Reasoning	Grasping mathematical concepts, ideas and relations; understanding multiple steps in complex procedures /algorithms; grasping basic logical principles (conditionality – "if…then…"statements – commutativity, inversion); grasping the semantic structure of problems; (strategic) decision-making; generalising.
Visual-spatial	Interpreting and using spatial organisation of representation of mathematical objects (for example, numbers in decimal position notation, exponents, geometrical 2D and 3D figures or rotations); placing numbers on number line; confusing Arabic numerals and mathematics symbols; performing written calculating when position is important (e.g. borrowing /carrying); interpreting graphs and tables.

We recall that the model framed also the design of Questionnaire B2, aimed at better understanding students' profiles of difficulty. When constructing B2, we chose questions that were related to the cognitive areas as well to three mathematical domains: arithmetic, geometry, algebra (core number is not related to all cognitive areas). As a result, we proposed questions that were located in some cells of the following table (Table 2).

Table 2: Double relation between cognitive areas (memory, reasoning and visuo-spatial) and mathematical domains (arithmetic, geometry, algebra).

	Arithmetic	Geometry	Algebra
Memory			
Reasoning			
Visual-spatial			

The same frame is used for the design of the intervention tools. Here we present additional theoretical references that support the design of the intervention tools.

First of all, we refer to the **Universal Design for Learning (UDL) principles** (Table 3), a framework specifically conceived to design *inclusive* educational activities (http://udlguidelines.cast.org/).

Table 3: UDL guidelines

	Provide multiple means of ENGAGEMENT	Provide multiple means of REPRESENTATION	Provide multiple means of ACTION and EXPRESSION
	Effective Networks – the "WHY" of learning	Recognition Networks – The "WAHT" of learning	Strategic Networks – The "HOW" of learning
Access	Provide options for **Recruiting Interest**: • Optimise individual choice and autonomy • Optimise relevance, value and authenticity • Minimise threats and distractions	Provide options for **Perception:** • Offers way of customising the display of information • Offer alternatives for auditory information • Offer alternatives for visual information	Provide options for **Physical Actions**: • Vary the method for response and navigation • Optimise access to tools and assistive technologies
Build	Provide options for **Sustaining Effort & Persistence**: • Heighten salience of goals and objectives • Vary demands and resources to optimise challenge • Foster collaboration and community • Increase master-oriented feedback	Provide options for **Language & Symbols**: • Clarify vocabulary and symbols • Clarify syntax and structure • Support decoding of text, mathematical notation and symbols • Promote understanding across languages • Illustrate through multiple media	Provide options for **Expression & Communication**: • Use multiple media for communication • Use multiple tools for construction and composition • Build fluencies with graduated levels of support for practice and performance
Internalise	Provide options for **Self- Regulation:** • Promote expectations and beliefs that optimise motivation • Facilitate personal coping skills and strategies • Develop self-assessment and reflection	Provide options for **Comprehension**: • Activate or supply backgrounds knowledge • Highlights pattern, critical features, big ideas and relationships • Guide information processing and visualisation • Maximise transfer and generalisation	Provide options for **Executive Function**: • Guide appropriate goal-setting • Support planning and strategy development • Facilitate managing information and resources • Enhance capacity for monitoring progress
	Expert Learners who are….		
	Purposeful & Motivated	Resourceful & Knowledgeable	Strategic and Goal-Directed

The Center for Applied Special Technology (CAST) has developed a comprehensive framework around the concept of Universal Design for Learning (UDL), with the aim of focusing research, development, and educational practice on understanding diversity and facilitating learning. UDL includes a set of Principles, articulated in *Guidelines and Checkpoints* (For a complete list of the principles, guidelines and checkpoints and a more extensive description of CAST's activities, visit http://www.udlcenter.org). The research grounding UDL's framework is that "learners are highly variable in their response to instruction. [...]"

Thus, UDL focuses on these individual differences as an important element to understand and design effective instruction for learning.

To this aim, UDL advances three foundational Principles: 1) provide multiple means of representation, 2) provide multiple means of action and expression 3) provide multiple means of engagement. In particular, guidelines within the first principle refer to means of perception involved in receiving certain information, and of "comprehension" of the information received. The guidelines within the second principle take into account the elaboration of information/ideas and their expression. Finally, the guidelines within the third principle deal with the domain of "affect" and "motivation", also essential in any educational activity.

For our analyses, we will focus in particular on specific guidelines within the three Principles (The items are taken from the interactive list at http://www.udlcenter.org/research/researchevidence).

Guidelines within Principle 1 (provide multiple means of representation), suggest proposing different options for perception and offering support for decoding mathematical notation and symbols. Moreover, guidelines suggest the importance of providing options for comprehension highlighting patterns, critical features, big ideas, and relationships among mathematical notions. Accordingly, we will propose the use of the software AlNuSet to guide information processing, visualization, and manipulation, in order to maximize transfer and generalization.

Moreover, the guidelines from Principle 2 (provide multiple means of action and expression) suggest to offer different options for expression and communication supporting planning and strategy development. Finally, the guidelines from

Principle 3 show how certain activities can recruit students' interest, optimizing individual choice and autonomy, and minimizing threats and distractions.
In section 4 we will present examples of activities, discussing the type of mathematical learning they address and the cognitive area they support. We will show how these examples have been designed within the frame of the UDL principles in order to make them inclusive and effective to overcame math difficulties identified through B2 questionnaire.

Another theoretical reference we refer to comes from the experience of the European Project **FasMed,** that focused on formative assessment in mathematics and science (https://research.ncl.ac.uk/fasmed/).

Formative assessment (FA) is conceived as a method of teaching where "evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited" (Black & Wiliam, 2009, p. 7). FaSMEd project refers to William and Thompson (2007)'s study, that identifies five key strategies for FA practices in school setting: (a) *clarifying and sharing learning intentions and criteria for success;* (b) *engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding;* (c) *providing feedback that moves learners forward;* (d) *activating students as instructional resources for one another;* (e) *activating students as the owners of their own learning.* The teacher, student's peers and the student him- or herself are the agents that activate these FA strategies. The FA strategies are summarized in table 4.

Table 4

	Where the learner is going	Where the learner is right now	How to get there
Teacher	**1** Clarify learning intentions and criteria for success	**2** Engineering effective classroom discussion and other learning tasks that elicit evidence of student understanding	**3** Providing feedback that moves learners forward
Peer	Understanding and sharing learning intentions and criteria for success	**4** Activating students as instructional resources for one another	
Learner	Understanding learning intentions and criteria for success	**5** Activating students as the owners of their own learning	

According to such conceptualization of formative assessment, the European Project FaSMEd designed and tested several class activities that exploit technology to support formative assessment strategies.

FaSMEd activities are organized in sequences that encompass group work on worksheets and class discussion where selected group works are discussed by the whole class, under the orchestration of the teacher. Taking into account formative assessment strategies and technology functionalities, Cusi, Morselli & Sabena (2017, p. 758) designed three types of worksheets for the classroom activity:

(1) *problem worksheets:* worksheets introducing a problem and asking one or more questions involving the interpretation or the construction of the representation (verbal, symbolic, graphic, and tabular) of the mathematical relation between two variables (e.g. interpreting a time-distance graph);
(2) *helping worksheets*, aimed at supporting students who face difficulties with *the problem worksheets* by making specific suggestions (e.g. guiding questions);
(3) *poll worksheets*: worksheets prompting a poll among proposed options".

The authors identified feedback strategies (Table 5) the teacher may adopt to give feedback to students (Cusi, Morselli & Sabena, 2018, p. 3466). These strategies are employed in the class discussion that is organized by the teacher after the group work on worksheets.

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Table 5: **Feedback Strategies**

Revoicing	When the teacher mirrors one student's intervention so as to draw the attention on it. Often, during the revoicing, the teacher stresses with voice intonation some crucial words of the sentence she is mirroring. Rephrasing takes place when the teacher reformulates the intervention of one student, with the double aim of drawing the attention of the class and making the intervention more intelligible to everybody.
Rephrasing	Rephrasing takes place when the teacher reformulates the intervention of one student, with the double aim of drawing the attention of the class and making the intervention more intelligible to everybody. Rephrasing is applied when the teacher feels that the intervention could be useful but needs to be communicated in a better way so as to become a resource for the others. [...] The revoicing and rephrasing strategies [...] turn one student (the author of the intervention) into a resource for the class.
Rephrasing with scaffolding	When the teacher, besides rephrasing, adds some elements to guide the students' work.
Relaunching	When the teacher reacts to a student's intervention, which (s)he considers interesting for the class, not giving a direct feedback, but posing a connected question. In this way, by relaunching the teacher provides an implicit feedback [...] on the student's intervention, suggesting that the issue is interesting and worth to be deepened or, conversely, has some problematic points and should be reworked on.
Contrasting	Contrasting takes place when the teacher draws the attention on two or more interventions, representing two different positions, so as to promote a comparison. By contrasting, [...] the authors of the two positions may be resource for the class as well as responsible of their own learning.

We draw from the FaSMEd experience the idea of creating classroom activities in the formative assessment perspective that may promote inclusion. An example of FaSMEd activity is the *Time-distance graphs activity*, that will be presented in section 4.

2. Designing the Intervention Tools – Guidelines

Once identified the areas of difficulties through the B2 questionnaire, it is possible to design educational activities especially conceived for students with Mathematical Learning Disabilities - MLD and with a specific attention on the inclusion.

In particular, the Intervention Tools can be:

- In the form of articulated activities that should be carried out with all the class, in a perspective of inclusion;
- In the form of specific exercises that could help a student having difficulties or all the students of the class to work together on the same activity.

In section 4, we present some examples of intervention tools focused on the following mathematical objects:

- **Mental calculation**, when difficulty is mainly related to the *Memory* cognitive area;
- **The meaning of variable and of expression depending on such a variable**, when difficulty is mainly related to the *Reasoning* cognitive area;

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- **Positional syntax in algebraic expressions**, when difficulty is mainly related to the *Visuo-spatial* cognitive area;
- **Interpretation of graphs**, when difficulty is mainly related to the *Visuo-spatial* cognitive area.

The following table (Table 6) presents the aforementioned mathematical objects and their connection with cognitive areas and mathematical domains.

Table 6: A first list of mathematical objects in the areas of difficulties identified through the B2 questionnaire.

	Arithmetic	Geometry	Algebra
Memory	**Knowing and applying procedures and strategies.** For instance, calculating mentally 36x11. An effective strategy to solve mentally this operation, requires the decomposition of 11 into 10+1 and the application of the distributive property, as in the following: 36x10=36 Then, you need to calculate the partial result of 36x10 (360) and add it to 36: 36x10+36=396. Students have to recover intermediate results.		
Reasoning			**Meaning of variable and of one expression in one variable** For instance, students may have difficulty in solving the following questions: - If a= 3 what is the value of 2a+1? - If x=4 what is the value of 24/x?
Visuospatial			Students may have difficulties in dealing with mathematical objects, due to their spatial representation. Mathematical objects that may be critical are for instance algebraic, expressions involving powers, because it is necessary to recognise the position of the symbol, since it changes its role in the expression according to its position. For instance, they may find difficult to distinguish between the following requests: - x^2 = - 2x = - x2 = Another mathematical object that is critical is the graph in the Cartesian plane.

As a general methodological comment, we point out that our aim is to design well-articulated teaching sequences, encompassing also group work and mathematical discussion under the orchestration of the teacher. The teaching sequences are conceived to address specific learning difficulty, within an inclusive perspective. The activities are not to be intended as mere exercises, unless they play the role of cognitive training. In cognitive training the student is led to perform a series of exercises that are focused in the same mathematical content, using ICT to have a repeated sequence.

In order to support the communication and sharing of intervention tools among partners and to all the teachers that may be interested into the project, we developed two templates for the presentation of the intervention tool.

1. Template IO1F is a table to be filled by the author of an intervention tool, providing all the basic information on the intervention tool.
2. Template IO1G contains the complete description of the intervention tool. In order to be auto-consistent, it should also contain the theoretical references that frame the design and implementation of the tool.

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The templates are presented hereunder:

IO1.F – DESCRIPTION OF THE INTERVENTION TOOL	
Title of the intervention tool*	*Please write down the title of the intervention tool*
The activity is conceived for*	☐ Individual ☐ Class
Cognitive area and mathematical domain addressed* *You can select only one choice*	☐ Core Number ☐ Memory/Arithmetic ☐ Memory/Geometry ☐ Memory/Algebra ☐ Reasoning/Arithmetic ☐ Reasoning/Geometry ☐ Reasoning/Algebra ☐ Visual-spatial/Arithmetic ☐ Visual-spatial/Geometry ☐ Visual-spatial/Algebra
Universal Design for Learning principles* *You can select more than one choice*	☐ Engagement/recruiting interest ☐ Engagement/Sustaining efforts and persistence ☐ Engagement/Self-regulation ☐ Representation/Perception ☐ Representation/Language and symbols ☐ Representation/Comprehension ☐ Action and expression/Physical action ☐ Action and expression/Expression and communication ☐ Action and expression/Executive functions
Formative assessment strategies* *You can select more than one choice*	☐ Clarifying learning intentions and criteria for success ☐ Engineering classroom discussions ☐ Providing feedback ☐ Activating students s resources for one another ☐ Activating learners as the owners of their own learning
Equipment needed*	*Please write down the equipment* (computer, tablet, projector, specific software…) *needed to carry out the intervention tool*
Estimated time*	*Please write down the estimated time needed to carry out the intervention tool*
Description*	*Please write down a short description of the intervention tool*
Educational Aim*	*Please indicate the educational aim of the intervention tool*
References*	*Please provide existing resources that inspired the planning of the intervention tool*
Annex	*Please produce a description of the practical activities of the Intervention Tool using the template IO1.G - Intervention Tool.*

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IO1.G– INTERVENTION TOOL
Introduction In this section, a general introduction to the intervention tool is presented. **Theoretical framework of reference** In this section, the common theoretical framework of reference is presented. **Design** In the subsections, the activities of the intervention tool are presented in detail: • **Difficulties identified through the B2 questionnaire** The intervention tool is presented in reference to a specific difficulty that was detected by means of the questionnaire. • **Cognitive area and math domain of interest** The specific difficulty that is mentioned in subsection 3.1 is to be linked to a cognitive area and mathematical domain of interest. • **Educational Aims** Once identified the difficulty, the intervention tool should aim at addressing such a difficulty. • **Addressing to Student /class** The intervention tool may be addressed to all the class or to single student. • **Educational activities: the Intervention Tool** In this subsection the activities are to be described in detail. **References** Reference for the theoretical framework are already provided.

3 Examples of Intervention Tools

Intervention Tool 1

We present an intervention tool that may useful in reference to difficulties highlighted in the following item of B2, Q3Al1 & Q3Al2:

If a=3 what is the value of 2a+1?

If x= -4, what is the value of 24/x?

As we already pointed out, difficulty in such item may be linked to the cognitive domain of *Reasoning* and in the domain of *Algebra*. The intervention tool is aimed at **Constructing the Meaning** (We point out that this does not just mean calculating the value of expressions nor manipulating algebraic expressions!) of variable and of expression in one variable.

Here we present a series of educational activities designed for the class.

The design of such activities relies on the use of UDL principles in order to make activities inclusive. In particular, we provide multiple means of representation, which promote both student's engagement and their action and expression.

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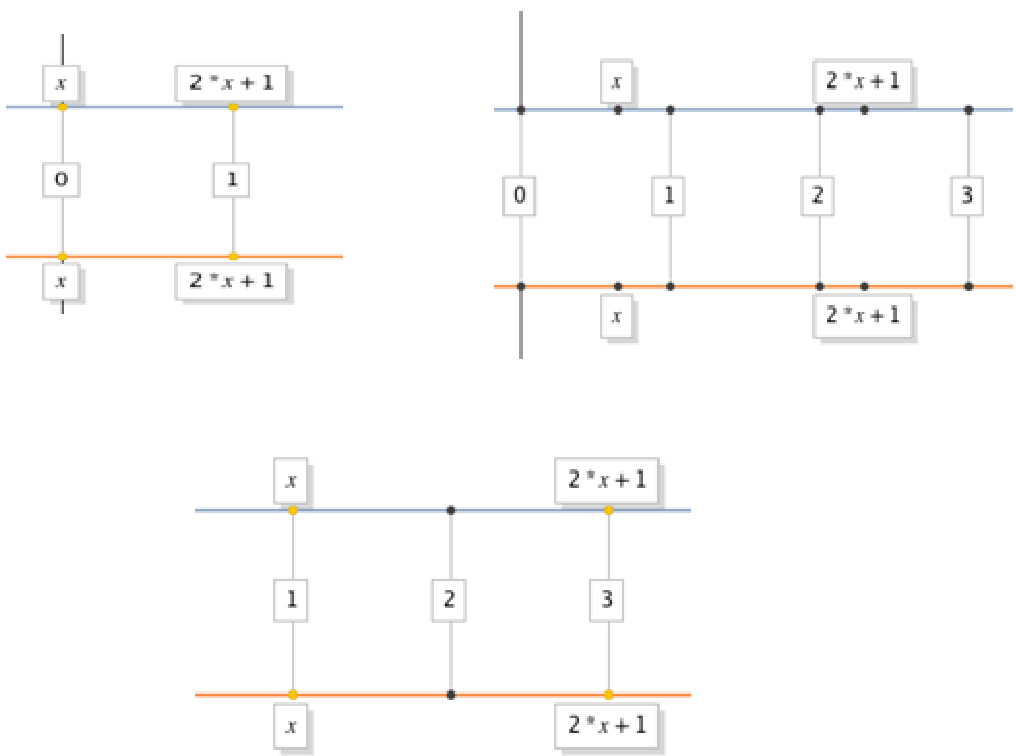
1) Dynamic representation of variable and expression depending on such a variable.

The first idea in designing activities relies on the use of the software AlNuSet, (see http://www.alnuset.com/en/alnuset). AlNuSet was designed for secondary school students (from age 12-13 to age 16-17) and it is made up of three separate environments that are tightly integrated: the Algebraic Line, the Algebraic Manipulator, and the Cartesian Plane. We will describe the features of the Algebraic Line, through the following activity (For a more detailed description of these environments see www.alnuset.com), which support the conceptualization of algebraic notions of variable and expression depending on a variable in MLD students (Robotti, E. 2016; Robotti E., Baccaglini-Frank A., 2017).

On the Algebraic Line it is possible to place variables and expressions that depend from them. To do this, the user has to type a letter, for example, “x”, and a mobile point will appear on the line. The point can vary within the chosen set of numbers (natural, whole, rational, or real - of course the representations of the numerical sets are accomplished on a computer, so the sets are actually finite and discrete, but they simulate – with some limitations – the properties of the number sets they represent.) and variation can be controlled directly by the user through dragging. This feature was designed so that important aspects of the notion of *variable* could become embodied. Moreover, it is possible to construct expressions on the line that depend on a chosen variable, for example, 2x+1. This dependent expression cannot be acted upon directly, but it will move as a consequence when x is dragged. The dependent expression will assume the positions on the line that correspond to the values it takes on when the dependent variable takes on the value it is dragged to (Figure 1).

Figure 1 represents the movement of the variable x on the Algebraic Line produces the movement of the dependent expression 2x+1 on the line.

Figure 1:



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We note that the functionalities described propose different representations (UDL Principle 1) and they are designed to foster for the user a mediation of the algebraic concepts of *variable* and *dependent expression,* through a dynamic model that can be acted upon (UDL Principle 2). The mediation can occur thanks to visual and kinaesthetic channels, without the need of visual verbal means (written language). The construction of the concept realized as so may allow students, and especially students with MLD, to find mnemonic references that are appropriate for their cognitive style. This allows them to start using representations of the fundamental algebraic concepts at stake, and possibly to place and retrieve them from long term memory in a more effective way.

With the support of AlNuSet, the teacher can promote a discussion among the students of the class in order to conceptualize the idea of *variable*.

As matter of fact, he/she can ask the students to move *x* along the line and to answer the following questions: "What can you observe?" "How do you interpret what happens?"

Moreover, the teacher can promote also a discussion among the students in order to conceptualize the idea of *expression depending on the variable x*.

Therefore, the teacher asks the students to digit *2x+1* in editor space of the Algebraic Line and he/she launch a discussion by the following question: "What happens on the Algebraic Line?"

"How do you interpret what happens to the algebraic expression 2x+1?"

It could be interesting, in a first time, to promote the definition of a hypothesis without the dynamic support of AlNuSet.

Thus, the teacher could ask the students: "If x=3, what do you think will be the value of the expression 2x+1? Make your hypothesis, compare it with your schoolmates and then verify it on the Algebraic Line of AlNuSet".

A discussion (guided by the teacher) about what students observe on the Algebraic Line and how they can interpret it in algebraic way, allows students to construct the meaning of *variable* and of *expression depending on such a variable*.

2) Representation of the relation between variable and expression depending on such a variable on a Cartesian plane and on a table

We consider a table defining the relation between the variable "x" and the expression 2x+1.

Table 7

x	2x+1
1	
2	
3	
0	
-1	
-4	

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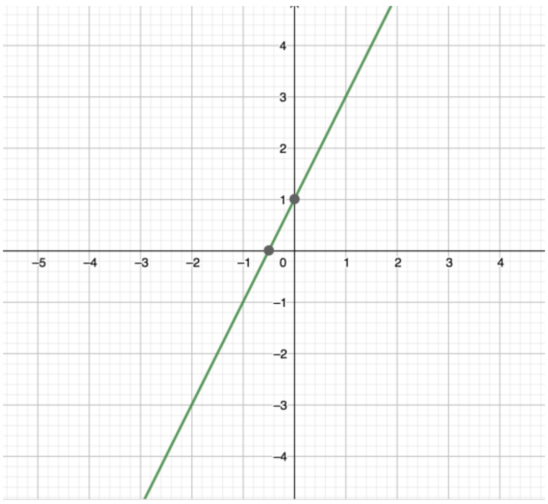
The teacher asks the students to calculate the value of the expression 2x+1 starting from the values of the independent variable "x":

Table 8

x	2x+1
1	2*1+1=2+1=3
2	2*2+1=4+1=5
3	2*3+1=6+1=7
0	2*0+1=0+1=1
-1	2*(-1)+1=-2+1=-1
-4	2*(-4)+1=-8+1=-7

The teacher asks students to draw the relation on the Cartesian plane:

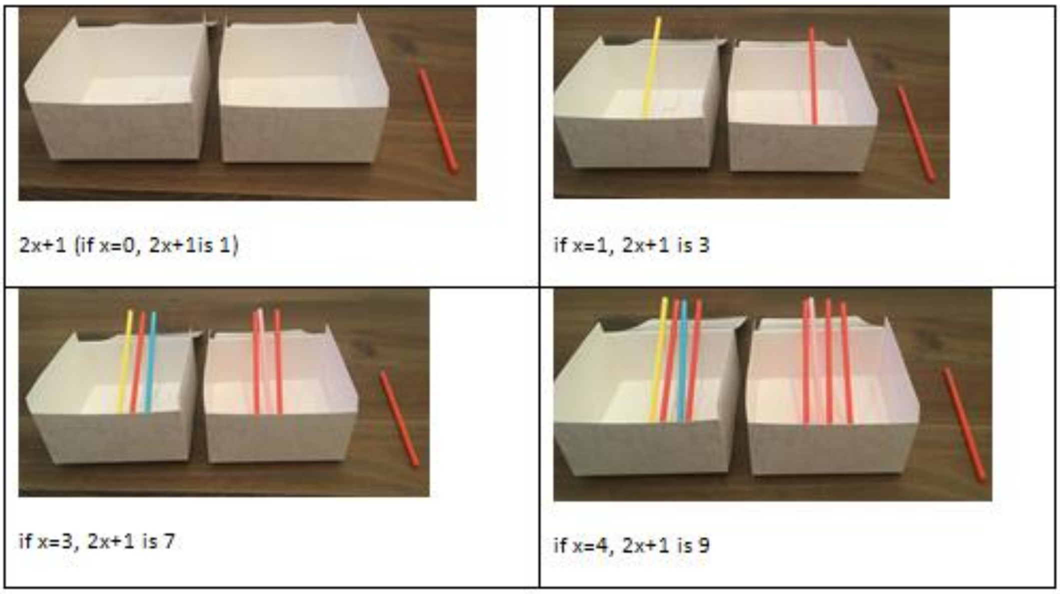


The teacher guides the discussion about the relation between *x* and the expression *2x+1* both through geometrical representation (on the Cartesian plane) and the algebraic relation (on the table) so that students will be able to pass from a code to the other one (transcoding process).

3) Concrete representation of a variable and of an expression depending on such a variable

The teacher presents two identical boxes (each represents *x*) and 1 straw (the constant), (Figure 2). By varying the number of straws in the boxes (the same for both, this means varying the value of the variable), the total straws vary (varying the value of the expression depending on such a variable).

Figure 2: Varying the value of the expression 2x+1 by varying the number of straws in the boxes (x).



The meaning of “variable” and of “expression which depends on such a variable” in algebra is constructed in a perceptive way by the manipulation of concrete objects.

Discussion through UDL guidelines about the above-mentioned activities

We observe that the same educational aim of constructing the meaning of “variable” and of “expression depending on such a variable” in algebra is approached in different ways by acting on the three principles of UDL (Table 7, in *red* our comments to illustrate the connection between the principles and our activities).

Table 9: Analysis of the activities through the Table of UDL principles.

Engagement	*Representation*	*Action & Expression*
Recruiting interest Optimize individual choice and autonomy Optimize relevance, value, and authenticity Minimize threats and distractions	***Perception*** Offer ways of customizing the display of information Offer alternatives for auditory information Offer alternatives for visual information Different registers through which information is displayed (visual-dynamic; visual; symbolic)	***Physical Action*** Vary the methods for response and navigation Optimize access to tools and assistive technologies
Sustaining effort Persistence Heighten salience of goals and objectives Vary demands and resources to optimize challenge Foster collaboration and community Increase mastery-oriented feedback Vary demands and resources to optimize challenge Foster collaboration and community Oriented feedbacks support engagement and motivation with respect the elaboration of the solution of the task	***Language & Symbols*** Clarify vocabulary and symbols Clarify syntax and structure Offer alternative language and symbols to decode information and to work on the information Support decoding of text, mathematical notation, and symbols *This is promoted by the dynamic action, and by the manipulation of objects* Promote understanding across languages Illustrate through multiple media *This is promoted by the activities of transcoding among different register of representation* Support decoding of text, math notation and symbols *This is promoted by the visualization of different registers at the same time (for example, on the Algebraic Line, a variable is a mobile point on the line and it is labelled by x)*	***Expression Communication*** Use multiple media for communication Use multiple tools for construction and composition **To use different registers in order to communicate** ***This is promoted by the use of terms that are alternative to the formal ones to speak about mathematical objects. Such alternative terms recall the meaning that was constructed by the students. For instance, students who worked with AlNuSet are keen to speak of "moving point" when they refer to the variable.*** *Moreover, in the activities virtual or concrete mathematical manipulatives are provided. For instance, dragging a moving point may help visualizing that the variable may have different values on the number line.* *Some activities that are connected to this principle are:* *- asking to read a table using AlNuSet (to transcode from table to AlNuSet)* *-asking to read AlNuSet with a table (to transcode AlNuSet into table)*
Self Regulation Promote expectations and beliefs that optimize motivation Facilitate personal coping skills and strategies Develop self-assessment and reflection *Formative assessment strategies, as discussed in section 2, may help self-assessment and reflection. More specifically, the teacher may provide different types of feedback*	***Comprehension*** Activate or supply background knowledge Highlight patterns, critical features, big ideas, and relationships (checkpoint 3.2) Guide information processing and visualization Maximize transfer and generalization Perception, language and symbols, comprehension (**Constructing useable knowledge, knowledge that is accessible for future decision-making, depends not upon merely perceiving information, but upon active "information processing skills")**	***Executive functions*** *Guide appropriate goal-setting* *The use of artefacts may also be a support for memory. Artefacts guide students' process of inquiry, providing feedback to their process.* *Support planning and strategy development* *Facilitate managing information and resources* *Enhance capacity for monitoring progress*

This allows students to construct meaning for the algebraic notions at stake.

Intervention Tool 2

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This intervention tool is conceived to address difficulties that may emerge when dealing with algebraic representation, in the considered case, the difficulty is more linked to the visuo-spatial cognitive domain than to the reasoning domain (see, for instance, Q4Al1, Q4Al2 and Q4Al4 of B2) .

Let's consider, for instance, the following algebraic expressions:

x^2 = …

2x = …

We may note that visuo-spatial difficulties may increase in advanced arithmetic, in comparison to elementary arithmetic which is based on the positional system of representation and only one direction (left-right) is involved. In advanced arithmetic other directions emerge: vertical position (fractions), oblique position (powers, roots, subscript). Moreover, symbols are written with different sizes, and different size and position convey a different meaning. Consider, for instance, the following expressions:

2; 22; –2; ½; 2^2; $2\sqrt{22}$; $2\sqrt{2}$

Moreover, when dealing with negative numbers it is necessary to conceive the minus sign as part of the number, and no more as an operating sign.

All the aforementioned facts may cause cognitive conflict in students, since it is necessary to reconstruct what was previously learnt in reference to natural numbers. Moreover, the student needs to perform a more complex visual synthesis of the expressions (both numerical and algebraic expressions). For instance, dealing with the following expressions requires connecting 2 and x in the proper way, depending on the position of the symbols.

x^2 = …

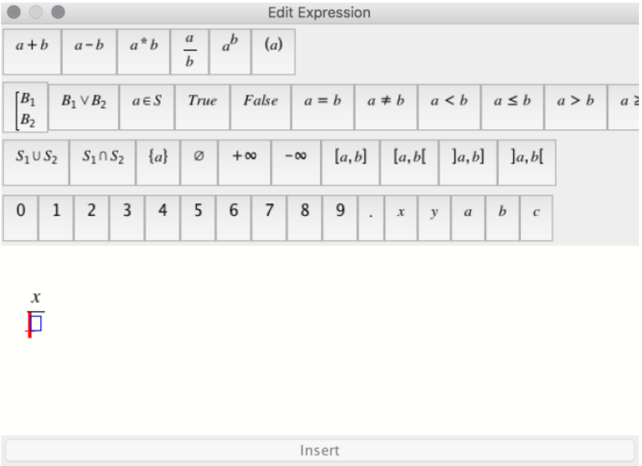
2x = …

This means that it is necessary to identity the structure of the expression in order to get its meaning. The structure can be outlined for instance by means of the Equation Editor of Word (Figure 3):

Figure 3: Equation Editor of Word to visualize structure of the expression

The same can be done by means of other editors, (Figure 4, or for instance, the editor of AlNuSet):

Figure 4: editor of AlNuSet



Intervention Tool 3

This intervention tool is conceived to address difficulties that may emerge when dealing with mental calculation (see, for instance, Q1.4 of B2 questionnaire).

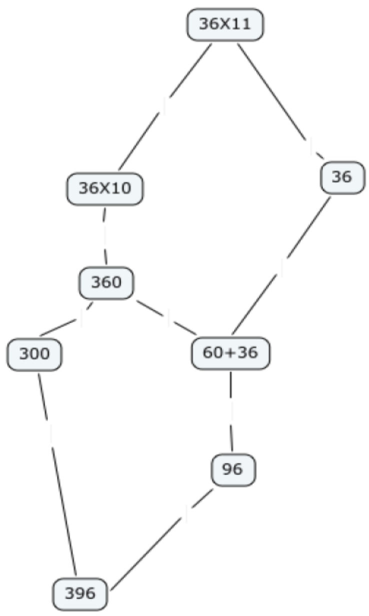
For example, in the case of the calculation:

36×11

Mental calculation requires an efficient management of executive functions, that could be slowed down by the need of keeping in mind intermediate results. If this happens, the whole calculation process risks to fail. In this case, we may say that the difficulty is not in the *knowledge* of mental calculation strategies, rather in *memory*: the student fails because of difficulty in keeping in mind and recovering the intermediate results of calculation.

The intervention is aimed at providing students with some support for memory. Representation systems that are efficient and fast in supporting the memorization and recovery may be useful. For instance, consider the following non-formal representation (Figure 5):

Figure 5: Example of informal writing as a support for the calculation process.



Intervention Tool 4

This intervention tool is conceived to address difficulties that may emerge when dealing with graphs in the Cartesian plane and that are linked to the Visuo-spatial cognitive domain (see, for instance, Q4Ar3, Q4Ar4 and Q4Ar5 of B2 questionnaire).

This intervention tool draws from the FaSMEd experience, (see https://microsites.ncl.ac.uk)

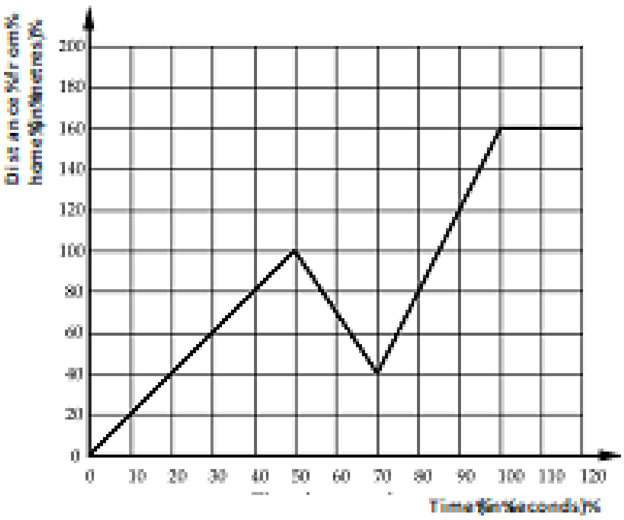
The intervention tool consists in guiding step-by-step the students into the interpretation of the graph and in giving large space to groupwork and class discussion, so that students act as resource for those mates that have more difficulty. Class discussion is also the occasion for the teacher to give specific feedback to students.

Here is a brief account of the sequence. Each question (worksheet, in the terminology of FaSMEd project) is to be administered to the students for the groupwork; after each question, a class discussion is orchestrated by the teacher.

Worksheet 1 introduces the graph and the corresponding story: the graph represents the way in which a student, Tommaso, walks, on a straight road, from home to the bus stop. The question posed to students makes them focus on the second section of the graph, that is the segment that connects the points (50, 100) and (70, 40). Students are asked to deduce, from the graph, what happens during the period of time from 50s to 70s.

Figure 6: Worksheet 1

Evert morning Tommaso walks along a straight road from is home to a bus stop a distance of 60 meters. The graph shows his journey in one particular day.



(1) What happens in the period of time between 50s and 70s? How do you know it?

We point out that students are asked to explain how they deduced this information from the graph, in order to make them reflect on the reasons supporting the correct interpretation of a time-distance graph.

Worksheet 1A is a helping worksheet, that can be provided to those students that have difficulty in answering to Worksheet 1. The teacher may decide to provide the helping worksheet to all those students that have difficulties linked to the visuo-spatial cognitive area.

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Figure 7: Worksheet 1A

(1)What happens in the period of time between 50s and 70s? How do you know it?

Help to answer question 1:

Remember that Tommaso is walking on a straight road.

- What is his distance from home after 50s?
- What is his distance from home after 70s?

The "help" within *worksheet 1A* aims at supporting the students in the interpretation of the graph in two ways:

1. the suggestion within the worksheet ("Remember that Tommaso is walking on a straight road") aims at preventing students from confusing the graph with the drawing of the road (proposing interpretations such as "Tommaso turns right, then left" or "Tommaso is down hill and then up again").
2. the two questions make the students focus on the way in which Tommaso's distance from home varies, helping the students observe that, since the distance is decreasing, Tommaso is approaching home.

Worksheet 1B is a worksheet prompting a poll: three answers, given by other fictitious students, are proposed, with the request of identifying the correct one.

Figure 8: Worksheet 1B

(1)What happens in the period of time between 50s and 70s? How do you know it?

What is the correct answer?

(a) In the period from 50s to 70s, Tommaso comes back.

(b) In the period from 50s to 70s, Tommaso changes his road

(c) In the period from 50s to 70s, the road, on which Tommaso is walking goes down.

Worksheet 2 shifts the focus on the last section of the graph, that is the horizontal segment (100,160)-(120,160).

Figure 9: Worksheet 2

(2)What happens during the last 20s? How did you establish it?

The question in *Worksheet 2* is focused on the interpretation of a horizontal line within a time-distance graph.

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Worksheet 3 requires students to determine when Tommaso reaches the bus stop. Here the focus is on the interpretation of a point in a time-distance graph as a bearer of two information: the distance from home and the time spent. Students have to identify the point (100,160) as the one on which they have to focus in order to find the answer.

Figure 10: Worksheet 3

(2)What happens during the last 20s? How did you establish it?

(a) After 120s

(b) After 50 + 70 + 100 + 120 second, that is after 340 seconds

(c) After 100 seconds

(d) After 50 seconds

The question in *worksheet 3* is proposed as a poll. The first option represents one of the typical mistakes made by students, who interpret the last point on the right of the graph as the one representing the moment in which Tommaso stops. The second option was inserted to see if students would have chosen it because of the "mathematical expression" proposed, without analysing its correctness. This poll is conceived as a starting point for a discussion focused on the reason underlying the choice of the answers.

Worksheet 4 is the last question proposed to support students' interpretation of the graph. It makes students focus on the distance walked by Tommaso to reach the bus stop.

Figure 11: Worksheet 4

(3)Does Tommaso walk for 160m? Why?

The question aims at making the students reflect one the difference between two concepts: the distance from home and the distance that was walked through. Again, students are also asked to share the reasons underlying their answers.

Worksheet 4A is a helping worksheet to be sent to those students who have difficulties in facing worksheet 4.

Figure 12: Worksheet 4A

(4)Does Tommaso walk for 160m? Why?

<u>Help to answer to question 4:</u>

Analyse the graph and answer to the following questions:

Figure 7

(a) What is the distance that Tommaso walked during the initial 120s?	Answer:
(b) What is the distance that Tommaso walked in the period from 50s to 70s?	Answer:
(c) What is the distance that Tommaso walked in the period of time from 70s to 100s?	Answer:
(d) What is the distance that Tommaso walked during the last 20s?	Answer:

Answer to question 4:

The "help" within *worksheet 4* consists of four different questions through which students are guided to focus, separately, on the different sections of the graph. In this way, they can determine the distance walked by Tommaso as the sum of the distances walked by Tommaso during the periods of time corresponding to each section of the graph.

Worksheet 5 focuses on a global interpretation of the graph. Students are asked to propose a possible completion of the story, in tune with the interpretation of the graph that the previous worksheets supported.

Figure 13: Worksheet 5

(5)After having answered to the questions in the previous worksheets, describe how Tommaso has walked on the road from his home to the bus stop. What coud have happened to him?

Worksheet 5 is aimed at enabling the students to recall all the aspects highlighted in the previous worksheets and corresponding discussions.

Discussion through UDL guidelines about the above-mentioned activities

We observe that the aforementioned teaching sequence is coherent with the three principles of UDL, as evidenced in the following table (Table 8, in *red* our comments to illustrate the connection between the principles and our activities).

Table 8: Analysis of the activities through the Table of UDL principles.

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Recruiting interest	Perception	Physical action
Sustaining effort and persistence: Group work and class discussion are functional to the aim of fostering collaboration and community. During class discussion the teacher and the peers may provide mastery-oriented feedback.	Language and symbol The worksheets may "Allow for flexibility and easy access to multiple representation of notation (e.g. formulas, word problems, graphs)"	Expression and communication Group work and class discussion may be efficient in "providing differentiated mentors (i.e. teachers/tutors who use different approaches to motivate, guide, feedback or inform)", "provide differentiated feedback (e.g. feedback that is accessible because it can be customised to individual learners). Helping worksheets mey be efficient "in providing scaffolds that can be gradually released with increasing independence and skills"
Self-regulation: Teachers and peers' feedbacks may promote subsequent self-regulation.	Comprehension: Analysing the graph step by step is a way of guiding the information processing and visualisation	Executive functions

References

[1] Black, P., & Wiliam, D. (2009). Developing the theory of formative assessment. Educational Assessment, Evaluation and Accountability, 21(1), 5-31.

[2] Cusi, A., Morselli, F.,& Sabena, C. (2017). Promoting formative assessment in a connected classroom environment: design and implementation of digital resources. Vol. 49(5), 755–767. ZDM Mathematics Education.

[3] Cusi, A., Morselli, F.,& Sabena, C. (2018). Enhancing formative assessment in mathematical class discussion: a matter of feedback. Proceedings of CERME 10, Feb 2017, Dublin, Ireland. hal-01949286, pp. 3460-3467.

[4] Karagiannakis, G. N., Baccaglini-Frank, A. E., & Roussos, P. (2016). Detecting strengths and weaknesses in learning mathematics through a model classifying mathematical skills. Australian J. of Learning Difficulties, 21(2), 115–141.

[5] Robotti E., Baccaglini-Frank A., (2017). Using digital environments to address students' mathematical learning difficulties. In *Innovation & Technology*. Series Mathematics Education in the Digital Era, A. Monotone, F. Ferrara (eds), Springer Publisher.

[6] Robotti E., (2016). Designing innovative learning activities to face up to difficulties in algebra of dyscalculia students: how exploit the functionality of AlNuSet. In *Digital Technologies in Designing Mathematics Education Tasks - Potential and pitfalls*. A. Baccaglini-Frank, A. Leung (eds), Springer Publisher.

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